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The existence of smooth solutions in a problem of the optimal control of the rotation of an axisymmetric rigid body☆

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ABSTRACT

The problem of the existence of a solution in the problem of the optimal control of the rotation of an axisymmetric rigid body for the arbitrary case of angular velocity boundary conditions is studied. A square integrable functional, which is consistent with the symmetry of the rotating body and characterizes the power consumption, is chosen as the criterion. The principal moment of the applied external forces serves as the control and the time of termination of a manoeuvre can be both specified as well as free. In the case of a specified termination time, it is shown that the solution (control) belongs to the class of infinitely-differentiable functions of time. The reasoning is based on the use of the singularities of the structure of the differential equations and the possibility of reducing the initial problem to two successive variational problems. The existence of a solution of the first of these problems in the class of square integrable functions is proved using the Cauchy–Bunyakovskii inequality. The second problem reduces to a search for the minimum of a functional which is weakly lower semi-continuous on a weakly compact set and the existence of its solution in the same class of functions follows from the Weierstrass theorem. The required conclusion concerning the smoothness of the solution of the optimal control problem is obtained from the necessary conditions of Pontryagin's maximum principle. In the case of a free termination time, one of the minimizing sequence can be constructed and it can be shown that, in the general case, there is no solution in the class of measurable controls.

The problem of the optimal control of the motion of an axisymmetric body under the assumption that a piecewise-continuous solution exists was considered in Ref. 1.

Because of the extreme non-linearity of the system, there are few results relating to the problem of the existence of solutions and the correctness of the determination of the corresponding class of functions for the family of variational problems studied, which also give rise to the need to introduce of the above mentioned assumption. The correctness of the assumption used concerning the smoothness of the optimal control is successfully proved and it is successfully strengthened. Moreover, a conclusion, similar to that obtained earlier,[2] that there is no solution of the problem with an unspecified termination time, is formulated.

1. Formulation of the problem

We shall study the problem of the optimal control of the spin up/spin down of an axisymmetric rigid body. The principal moment of the external forces applied to the body is used as the control. The change in the angular velocity vector from the initial value to the required terminal value in such a way that the manoeuvre corresponds to the least energy consumption is assumed to be the basic task of the control A square integrable functional, which is consistent with the symmetry of the rotating body, is adopted as the criterion. The boundary conditions for the angular velocity vector can be arbitrary and the change in the orientation will be ignored. The cases of specified and free termination times are considered.

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Thus, the optimal control problem[1]

$$\int\limits_{[0,T]} (u_1^2(t) + u_2^2(t) + C^{-1}u_3^2(t))dt \to \min_{u_1, u_2, u_3}, \quad t \in [0, T]$$

$$\omega_1(0) = \upsilon_1, \quad \dot{\omega}_1 = K\omega_2\omega_3 + u_1 \quad \text{(a.e.)}, \quad \omega_1(T) = w_1$$

$$\omega_2(0) = \upsilon_2, \quad \dot{\omega}_2 = -K\omega_1\omega_3 + u_2 \quad \text{(a.e.)}, \quad \omega_2(T) = w_2$$

$$\omega_3(0) = \upsilon_3, \quad \dot{\omega}_3 = u_3 \quad \text{(a.e.)}, \quad \omega_3(T) = w_3$$

$$\mathbf{u} = (u_1, u_2, u_3)^T, \ \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T, \ \boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \upsilon_3)^T, \ \mathbf{w} = (w_1, w_2, w_3)^T, \ C > 0 \quad (1.1)$$

is studied, where $\boldsymbol{\omega}$ is the angular velocity vector in projections onto the associated system of coordinates, the axes of which coincide with the principal central axes of inertia of the body, $\mathbf{u}$ is the control vector which is connected to the vector of the principal moment of the external forces $\mathbf{M} = (M_1, M_2, M_3)^T$ by the relations

$$u_i = I_i^{-1}M_i, \quad i = 1, 2, 3; \quad I_1 = I_2, \quad K = 1 - I_3/I_1$$

and $I_i > 0$ are the principal central moments of inertia. Here and henceforth a.e. denotes “almost everywhere with respect to the classical Lebesgue measure” and, unless otherwise stated, it is borne in mind that $t \in [0, T]$.

One of the serious difficulties, which is mostly eliminated by introducing an additional assumption, lies in the indication of the class of functions in which a solution of the optimal control problem exists and in the strict proof of the fact that a solution of the problem exists. These questions, as applied to the problem of the rotation of an axisymmetric body, will be discussed later.

Everywhere, apart from in the last section, we shall assume that the termination time $T > 0$ is specified. We now introduce some notation. Suppose $\mathcal{A}$ is a set of the form

$$\mathcal{A} = \{u_1, u_2, u_3 \in L_2\colon \omega_1(0) = \upsilon_1, \dot{\omega}_1 = K\omega_3\omega_2 + u_1, \text{(a.e.)}, \omega_1 T = w_1;$$
$$\omega_2(0) = \upsilon_2, \dot{\omega}_2 = -K\omega_3\omega_1 + u_2\text{(a.e.)}, \omega_2(T) = w_2; \omega_3(0) = \upsilon_3, \dot{\omega}_3 = u_3\text{(a.e.)},$$
$$\omega_3(T) = w_3\}$$

$J(\mathbf{u}) : L_2 \times L_2 \times L_2 \to \mathbb{R}$ is a functional of the form

$$J(\mathbf{u}) = \int (u_1^2(t) + u_2^2(t) + C^{-1}u_3^2(t))dt$$

and the integrals are understood in the Lebesgue sense using the classical measure. Henceforth, $L_2 = L_2([0, T])$ and, unless otherwise stated, integration is carried out over the interval $[0, T]$.

Generally speaking, it follows from Euler’s Eq. (1.1) that the functions u_i must be only Lebesgue integrable, that is, they belong to the class $L_1[(0, T])$. However, in order that the functional should take finite values, it is sufficient to restrict the space to $L_2([0, T])$, which is what has been done above.

The existence of a solution of the problem

$$\min_{\mathbf{u} \in \mathcal{A}} J(\mathbf{u}) \quad (1.2)$$

will therefore be studied.

It will be proved later that problem (1.2) has a solution. It will subsequently be proved that a solution of the initial problem exists in a narrower class, the class of infinitely-differentiable functions of time $C^\infty([0, T])$. In conclusion, it will be shown that, apart from rare cases of boundary conditions, there are no solutions of the problem with an unspecified time.

2. Subsidiary optimal control problems

Suppose $z \in \mathrm{AC}([0, T])$ is an arbitrarily chosen function, absolutely continuous in $[0,T]$ with boundary conditions $z(0) = v_3$ and $z(T) = w_3$, and $z \in L_1([0, T])$ is its (a.e.) derivative. For the specified function $\dot{z}$, we consider the optimal control problem

$$\int (u_1^2(t) + u_2^2(t))dt \to \min_{u_1, u_2}$$

$$\boldsymbol{\omega}_1(0) = \upsilon_1, \quad \dot{\boldsymbol{\omega}}_1 = Kz\boldsymbol{\omega}_2 + u_1 \quad \text{(a.e.)}, \quad \boldsymbol{\omega}_1(T) = w_1$$

$$\boldsymbol{\omega}_2(0) = \upsilon_2, \quad \dot{\boldsymbol{\omega}}_2 = -Kz\boldsymbol{\omega}_1 + u_2 \quad \text{(a.e.)}, \quad \boldsymbol{\omega}_2(T) = w_2 \quad (2.1)$$

for $u_1, u_2 \in L_2([0, T])$. We introduce the notation

$$\mathcal{A}(\dot{z}) = \{u_1, u_2 \in L_2([0, T]):$$
$$\boldsymbol{\omega}_1(0) = \upsilon_1, \dot{\boldsymbol{\omega}}_1 = Kz\boldsymbol{\omega}_2 + u_1(\text{a.e.}), \boldsymbol{\omega}_1(T) = w_1; \boldsymbol{\omega}_2(0) = \upsilon_2, \dot{\boldsymbol{\omega}}_2 = -Kz\boldsymbol{\omega}_1 + u_2(\text{a.e.}),$$
$$\boldsymbol{\omega}_2(T) = w_2\}$$
$$H(u_1, u_2) = \int(u_1^2(t) + u_2^2(t))dt$$

and write problem (2.1) more compactly as

$$\min_{(u_1, u_2) \in \mathcal{A}(\dot{z})} H(u_1, u_2) \tag{2.2}$$

We will assume that $B \in \mathrm{SO}(2)$ is the matrix of rotation in a plane, which is the solution of the Cauchy problem

$$B(0) = I, \quad \dot{B} = Kz\begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}B \tag{2.3}$$

where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix. In problem (2.1), we change to new variables, which are connected with the old variables by the relations

$$\begin{Vmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{Vmatrix} = B^T\begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}, \quad \begin{Vmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Vmatrix} = B^T\begin{Vmatrix} u_1 \\ u_2 \end{Vmatrix} \tag{2.4}$$

The correctness of the equalities

$$\dot{\tilde{\omega}}_1 = \tilde{u}_1, \quad \dot{\tilde{\omega}}_2 = \tilde{u}_2 \ (\text{a.e.}); \quad H(u_1, u_2) = H(\tilde{u}_1, \tilde{u}_2)$$

is immediately verified.

It is well known that the solution of problem (2.3) can be represented in the form

$$B(t) = \begin{Vmatrix} \cos\varphi(t) & \sin\varphi(t) \\ -\sin\varphi(t) & \cos\varphi(t) \end{Vmatrix}$$

where φ is an absolutely continuous function, which is the solution of the Cauchy problem

$$\varphi(0) = 0(\mathrm{mod}2\pi), \quad \dot{\varphi} = Kz \ (\text{a.e.})$$

and is completely determined by specifying the function z. The corresponding solution has the form

$$\varphi(t) = K\int_{[0, t]} z(\tau)d\tau(\mathrm{mod}2\pi)$$

Consequently, the formulae defining the boundary conditions for the new variables have the form

$$\begin{Vmatrix} \tilde{\omega}_1(0) \\ \tilde{\omega}_2(0) \end{Vmatrix} = \begin{Vmatrix} \upsilon_1 \\ \upsilon_2 \end{Vmatrix}, \quad \begin{Vmatrix} \tilde{\omega}_1(T) \\ \tilde{\omega}_2(T) \end{Vmatrix} = \begin{Vmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{Vmatrix} = \begin{Vmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{Vmatrix}\begin{Vmatrix} w_1 \\ w_2 \end{Vmatrix}, \ \chi = K\int_{[0, T]} z(t)dt$$

Using the substitution of the variables (2.4) in problem (2.1) and taking account of the inequalities for the boundary conditions presented above, we arrive at the problem

$$H(\tilde{u}_1, \tilde{u}_2) \to \min_{\tilde{u}_1, \tilde{u}_2}$$
$$\tilde{\omega}_1(0) = \upsilon_1, \dot{\tilde{\omega}}_1 = \tilde{u}_1 \ (\text{a.e.}); \tilde{\omega}_1(T) = \tilde{w}_1; \tilde{\omega}_2(0) = \upsilon_2, \dot{\tilde{\omega}}_2 = \tilde{u}_2 \ (\text{a.e.}), \tilde{\omega}_2(T) = \tilde{w}_2 \tag{2.5}$$

Solutions of problems (2.1) and (2.5) exist or do not exist simultaneously since the corresponding variables are related by the equalities (2.4). We therefore later change to the study of problem (2.5). The basic difference between problems (2.1) and (2.5) lies in the fact that the initial problem is a problem of the optimal control of a linear non-steady-state system, and problem (2.5), which is obtained after the change of variables, is an optimal control problem for a steady-state linear system of simpler structure. This property enables us to change later to the simplest linear-quadratic problem.

In fact, suppose

$$\mathcal{A}_i = \{\tilde{u}_i \in L_2: \tilde{\omega}_i(0) = \upsilon_i, \dot{\tilde{\omega}}_i = \tilde{u}_i \ (\text{a.e.}) \ t \in [0, T], \tilde{\omega}_i(T) = \tilde{w}_i\}, \quad i = 1, 2$$

We then rewrite problem (2.5) in the form

$$\min_{\substack{\tilde{u}_1 \in \mathcal{A}_1 \\ \tilde{u}_2 \in \mathcal{A}_2}} H(\tilde{u}_1, \tilde{u}_2) \tag{2.6}$$

and since, obviously,

$$\inf_{\substack{\tilde{u}_1 \in \mathcal{A}_1 \\ \tilde{u}_2 \in \mathcal{A}_2}} \int (\tilde{u}_1^2(t) + \tilde{u}_2^2(t))dt = \sum_{i=1}^{2} \inf_{\tilde{u}_i \in \mathcal{A}_i} \int \tilde{u}_i^2(t)dt \tag{2.7}$$

then, for problem (2.6) to be solvable, it is sufficient to show that an exact lower bound is attained for each problem on the right-hand side of equality (2.7).

Hence, we arrive at the following simplest scalar linear-quadratic optimal control problem

$$\min_{u \in \mathcal{D}} \int u^2(t)dt$$
$$\mathcal{D} = \{u \in L_2 \colon x(0) = a, \dot{x} = u(\text{a.e.}), x(T) = b\} \tag{2.8}$$

We will show that a unique solution exists in this problem and we will construct this solution.

In fact, we note that the set $\mathcal{D}$ can be written in the equivalent form

$$\mathcal{D} = \{u \in L_2 : \int u(t)dt = b - a \equiv c\} \tag{2.9}$$

It follows from the Cauchy–Bunyakovskii inequality that

$$\int u(t)dt \le \left(\int dt\right)^{1/2} \left(\int u^2(t)dt\right)^{1/2}$$

Since,

$$\left|\int u(t)dt\right| \le \int |u(t)| dt$$

then, from definition (2.9), we obtain the inequality

$$c^2 T^{-1} \le \int u^2(t)dt$$

which is true for all $u \in \mathcal{D}$. Consequently,

$$c^2 T^{-1} \le \inf_{u \in \mathcal{D}} \int u^2(t)dt$$

We now define the function

$$u^*(t) = cT^{-1} \quad (\text{a.e.}) \tag{2.10}$$

Then, obviously, $u^* \in \mathcal{D}$ and, since

$$\int (u^*(t))^2 dt = c^2 T^{-1}$$

we obtain the equality

$$\int (u^*(t))^2 dt = \inf_{u \in \mathcal{D}} \int u^2(t)dt$$

It has therefore been proved that a solution of problem (2.8) exists and one of the solutions has the form (2.10).

The solution of problem (2.8) is unique apart from to an equivalent function.

Actually, if $a = b$, the assertion is obvious and corresponds to a unique null control. Now suppose $a \neq b$. We shall reason by contradiction and assume that, in reality, $u^*, u^{**} \in \mathcal{D}$ are two different solutions (non-coinciding on a set of non-zero measure) of variational problem (2.8). It can then be concluded that the functions u^* and u^{**}, considered as vectors in the linear space L_2, are not collinear. Actually, if they were to be collinear, then the equality

$$u^*(t) = du^{**}(t) \quad (\text{a.e.})$$

would hold for a certain constant d and, therefore, $d^2 = 1$. Consequently, one of the two equalities

$$u^*(t) = u^{**}(t) \quad (\text{a.e.}), \quad u^*(t) = -u^{**}(t) \quad (\text{a.e.})$$

holds. Since $u^*, u^{**} \in \mathcal{D}$, it follows from representation (2.9) that, for the case when $a \neq b$, the first equality must be satisfied. Consequently, the solutions are identical almost everywhere with respect to the classical Lebesgue measure and this contradicts the assumption.

We now arbitrarily choose a real number α and construct the function $\alpha u^* + (1-\alpha)u^{**}$. It follows from formula (2.9) that this function belongs to the set $\mathcal{D}$, and we thereby obtain a single-parameter family of controls which are permissible in problem (2.8). The equality

$$\int(\alpha u^*(t) + (1-\alpha)u^{**})^2 dt = (\alpha^2 + (1-\alpha)^2)r + 2\alpha(1-\alpha)s$$

is obvious, where

$$r = \int(u^*(t))^2 dt = \int(u^{**}(t))^2 dt > 0, \quad s = \int u^*(t)u^{**}(t)dt$$

Again, using the Cauchy–Bunyakovskii inequality, we obtain

$$\left(\int u^*(t)u^{**}(t)dt\right)^2 \le \int(u^*(t))^2 dt \int(u^{**}(t))^2 dt$$

It is well known that the Cauchy–Bunyakovskii inequality is satisfied as an equality in and only in the case when the vectors are collinear. Since the non-collinearity of u^* and u^{**} has been proved earlier, then $s^2 < r^2$.

The equality

$$\min_{u \in \mathcal{D}} \int u^2(t)dt = \min_{\alpha}(\alpha^2 + (1-\alpha)^2)r + 2\alpha(1-\alpha)s$$

follows from the constructions. From the necessary condition for an extremum of the function on the right-hand side, we obtain

$$(r-s)(2\tilde{\alpha}-1) = 0$$

where $\tilde{\alpha}$ is the point of the unique extremum. Consequently, $\tilde{\alpha} = 1/2$ and, then,

$$\min_{u \in \mathcal{D}} \int u^2(t)dt = (\tilde{\alpha}^2 + (1-\tilde{\alpha})^2)r + 2\tilde{\alpha}(1-\tilde{\alpha})s = (r+s)/2 \le (r+|s|)/2 < r$$

The strict inequality means that u^* and u^{**} are not solutions of problem (2.8). The resulting contradiction proves the required uniqueness of the solution.

Apart from the proof of the existence and uniqueness of the solution of problem (2.8), the solution of (2.10) was obtained, which enables as to write the equality

$$\min_{u \in \mathcal{D}} \int u^2(t)dt = (a-b)^2 T^{-1}$$

The result presented can also be proved on the basis of other facts. For example, it is possible to make use of a theorem on the existence of a unique projection of a point onto a closed convex set in Hilbert space or to use a version of the Weierstrass theorem on the minimum of a weakly continuous functional on a weakly compact set. It is also possible to refer to the existence theorem for the general linear-quadratic problem (see Ref. 3, Ch. 16, §16.2).

Hence, the arguments presented above as applied to equality (2.7) lead to the relation

$$\min_{\substack{\tilde{u}_1 \in \mathcal{A}_1 \\ \tilde{u}_2 \in \mathcal{A}_2}} \int(\tilde{u}_1^2(t) + \tilde{u}_2^2(t))dt = \sum_{i=1}^{2} \min_{\tilde{u}_i \in \mathcal{A}_i} \int \tilde{u}_i^2(t)dt = T^{-1}\sum_{i=1}^{2}(\tilde{w}_i - \upsilon_i)^2$$

Using the corresponding formulae for $\tilde{w}_1$ and $\tilde{w}_2$, we introduce the notation

$$F(\dot{z}) = T^{-1}\sum_{i=1}^{2}(\tilde{w}_i - \upsilon_i)^2 = T^{-1}[(w_1\cos\chi - w_2\sin\chi - \upsilon_1)^2 + (w_1\sin\chi + w_2\cos\chi - \upsilon_2)^2] \tag{2.11}$$

for the functional $F : L_2 \to \mathbb{R}$. The final result now takes the form

$$\min_{(u_1, u_2) \in \mathcal{A}(\dot{z})} H(u_1, u_2) = F(\dot{z}) \tag{2.12}$$

3. Reduction of the initial problem to a problem of successive minimization

The arguments presented in this section are based on the following assumption.

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary spaces and $G : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a certain functional. In the case of the set

$$\mathcal{C} \subset \mathcal{X} \times \mathcal{Y}$$

we consider the sets

$$\mathrm{Pr}_{\mathcal{X}}\mathcal{C} = \{x \in \mathcal{X} : \exists y \in \mathcal{Y}, (x, y) \in \mathcal{C}\}, \quad \mathcal{C}(x) = \{y \in \mathcal{Y} : (x, y) \in \mathcal{C}\}$$

which have the meaning of a projection and a section of $\mathcal{C}$ respectively. The equality

$$\inf_{(x, y) \in \mathcal{C}} G(x, y) = \inf_{x \in \mathrm{Pr}_{\mathcal{X}}\mathcal{C}} \inf_{y \in \mathcal{C}(x)} G(x, y) \tag{3.1}$$

then holds.

In fact, the inequality

$$\inf_{(x,y)\in\mathscr{C}} G(x,y) \le \inf_{x\in \mathrm{Pr}_{\mathscr{X}}\mathscr{C}} \inf_{y\in\mathscr{C}(x)} G(x,y) \tag{3.2}$$

follows from the definition of the exact lower bound.

Suppose $\{(x_n, y_n)\}$ is the minimizing sequence in the problem

$$\inf_{(x,y)\in\mathscr{C}} G(x,y)$$

that is,

$$\lim_{n\to\infty} G(x_n, y_n) = \inf_{(x,y)\in\mathscr{C}} G(x,y)$$

Since $(x_n, y_n)\in\mathscr{C}$, , then $y_n\in\mathscr{C}(x_n)$ and, therefore,

$$G(x_n, y_n) \ge \inf_{y\in\mathscr{C}(x_n)} G(x_n, y)$$

Now, taking account of the fact that $x_n \in \mathrm{Pr}_x\mathscr{C}$, on the right-hand side we take the exact lower bound:

$$\inf_{y\in\mathscr{C}(x_n)} G(x_n, y) \ge \inf_{x\in \mathrm{Pr}_{\mathscr{X}}\mathscr{C}} \inf_{y\in\mathscr{C}(x)} G(x,y),$$

from which

$$G(x_n, y_n) \ge \inf_{x\in \mathrm{Pr}_{\mathscr{X}}\mathscr{C}} \inf_{y\in\mathscr{C}(x)} G(x,y)$$

On taking the limit with respect to n and taking account of the definition of the minimizing sequence, we obtain the inequality

$$\inf_{(x,y)\in\mathscr{C}} G(x,y) \ge \inf_{x\in \mathrm{Pr}_{\mathscr{X}}\mathscr{C}} \inf_{y\in\mathscr{C}(x)} G(x,y) \tag{3.3}$$

What is required then follows from inequalities (3.2) and (3.3).

We now consider the set $\mathcal{A}$ from Section 1 and put

$$\mathrm{Pr}_3\mathscr{A} = \{u_3 \in L_2 : \exists u_1, u_2 \in L_2, (u_1, u_2, u_3) \in \mathscr{A}\}$$

The representation

$$\mathrm{Pr}_s\mathscr{A} = \{u_3 \in L_2 : \omega_3(0) = \upsilon_3, \dot{\omega}_3 = u_3(\text{a.e.}), \omega_3(T) = w_3\} \tag{3.4}$$

is correct.

Actually, the inclusion

$$\mathrm{Pr}_3\mathscr{A} \subset \{\cdot\}$$

where $\{\cdot\}$ is the right-hand side of equality (3.4) is obvious by virtue of the definition of the set $\mathcal{A}$. Conversely, we choose an arbitrary function u_3 from the set $\{\cdot\}$. The inverse inclusion, which also proves the required equality, follows from the definition of the set $\mathcal{A}(\dot{z})$ for $\dot{z} = u_3$.

Hence, introducing the notation

$$J_{123} = \int (u_1^2(t) + u_2^2(t) + C^{-1}u_3^2(t))dt, \quad J_3 = C^{-1}\int u_3^2(t)dt$$

and applying equality (3.1) to the initial problem, we arrive at the formulae

$$\inf_{(u_1,u_2,u_3)\in\mathscr{A}} J_{123} = \inf_{u_3\in\mathrm{Pr}_3\mathscr{A}} \inf_{(u_1,u_2)\in\mathscr{A}(u_3)} J_{123} == \inf_{u_3\in\mathrm{Pr}_3\mathscr{A}} [J_3 + \inf_{(u_1,u_2)\in\mathscr{A}(u_3)} \int (u_1^2(t) + u_2^2(t))dt]$$

The second term in the square brackets is identical to the solution of the problem considered in Section 2. Using expression (2.12), we therefore obtain

$$\inf_{(u_1,u_2,u_3)\in\mathscr{A}} J_{123} = \inf_{u_3\in\mathrm{Pr}_3\mathscr{A}} [J_3 + F(u_3)] \tag{3.5}$$

The functional F is described by formula (2.11).

Hence, to prove the existence of a solution of the initial problem (2.1), it is sufficient to show that the exact lower bound on the right-hand side of equality (3.5) is attained.

4. The existence of an infinitely differentiable solution of the initial problem

The functional $\Phi : L_2 \to \mathbb{R}$, defined by the formula

$$\Phi(u) = C^{-1}\int u^2(t)dt + F(u) \tag{4.1}$$

is weakly lower semicontinuous.

In fact, taking account of the fact that $\dot{z} = u$ in formula (2.11), we obtain $z(t) = z(0) + \int_{[0,t]} u(\tau)d\tau$ and

$$\int\limits_{[0,T]} z(t)dt = z(0)T + \int\limits_{[0,T]}\left(\int\limits_{[0,t]} u(\tau)d\tau\right)dt \tag{4.2}$$

It is easily verified that the functional $f : L_2 \to \mathbb{R}$

$$f(u) = \int\limits_{[0,T]}\left(\int\limits_{[0,t]} u(\tau)d\tau\right)dt$$

is bounded and linear. Then, $f \in L_2^*$ and the functional f is weakly continuous by definition. The functional (4.2) is weakly continuous as the sum of a weakly continuous functional and a constant functional. Now, the functional F is weakly continuous as a continuous function of a weakly continuous functional.

The formula

$$g(u) = \int u^2(t)dt \equiv \|u\|_{L_2}^2 \equiv \|u\|^2$$

defines a continuous convex functional. This functional (and, consequently, also the first term on the right-hand side of equality (4.1)) is weakly lower semi continuous (see for example, Ref. 4, Proposition 3, p. 402). Consequently, the functional Φ is continuous as the sum of a weakly continuous functional and a weakly lower semi continuous functional.

We will now consider a closed fall $\mathfrak{b}_R$ of radius $R > 0$ with centre at the origin in the space L_2.

$$\mathfrak{b}_R = \{u \in L_2 : \|u\|_{L_2} \le R\}$$

For any $R > 0$, a solution of the following problem exists

$$\min_{u \in \mathrm{Pr}_3\mathcal{A} \cap \mathfrak{b}_R} \Phi(u) \tag{4.3}$$

In fact, we represent the set $Pr_3\mathcal{A}$ in the form

$$\mathrm{Pr}_3\mathcal{A} = \{u \in L_2 : \int u(t)dt = w_3 - \upsilon_3\} = \{u \in L_2 : \langle u, 1\rangle = w_3 - \upsilon_3\}$$

where $\langle\cdot,\cdot\rangle$ is the symbol of a scalar product in $L_2[(0, 1)]$. Consequently, the set $Pr_3\mathcal{A}$ is a hyperplane, and it is therefore closed and convex. We arrive at the conclusion that $\mathrm{Pr}_3\mathcal{A} \cap \mathfrak{b}_R$ is a closed, bounded, convex set. It is well known (Ref. 5, p. 146, Ref. 6, Theorem 7, p. 178, and Ref. 7, Corollary 14, p. 457) that a closed convex set is weakly closed. From the reflexiveness of the space L_2 and the theorem on the weak compactness (Ref. 4, Theorem 1, p. 397, and Ref. 6, Theorem 2, p. 201), we conclude that the set $\mathrm{Pr}_3\mathcal{A} \cap \mathfrak{b}_R$ is weakly compact. Finally, the required result follows from the strong version of the Weierstrass theorem (Ref. 4, Theorem 3′, p. 401, and Ref. 6, Corollary 2, p. 462).

The functional (4.1) satisfies the condition of increasing to infinity

$$\Phi(u) \to +\infty \text{ when } \|u\| \to \infty \tag{4.4}$$

Actually, by construction $F(u) \ge 0$, and, therefore,

$$\Phi(u) = C^{-1}\|u\|^2 + F(u) \ge C^{-1}\|u\|^2$$

and the required construction is obvious.

We now construct the function

$$u^0(t) = (w_3 - \upsilon_3)T^{-1}$$

It can be verified that $u^0 \in Pr_3\mathcal{A}$ and, consequently, $\Phi(u^0) < \infty$. According to condition (4.4) for u^0, a number $R^0 > 0$ is found such that

$$\Phi(u) \ge \Phi(u^0) \text{ when } \|u\| \ge R^0$$

Consequently,

$$\inf_{u \in \mathrm{Pr}_3\mathcal{A}} \Phi(u) = \min_{u \in \mathrm{Pr}_3\mathcal{A} \cap \mathfrak{b}_{R^0}} \Phi(u)$$

From the existence of a solution of problem (4.3), we now obtain that a solution of the problem

$$\min_{u \in \mathrm{Pr}_3 \mathcal{A}} \Phi(u) \tag{4.5}$$

exists.

Suppose u_3 is a certain solution of problem (4.5) and (u_1, u_2) is the corresponding solution of problem (2.12) for $\dot{z} = u_3$. Since, by construction $u^0 \in Pr_3\mathcal{A}$, $(u_1, u_2) \in \mathcal{A}(u_3)$, then $(u_1, u_2, u_3) \in \mathcal{A}$. The existence of a solution of the initial problem (1.2) now follows from equality (3.5).

Note that the following fact has been incidentally proved: $(u_1, u_2, u_3)^T$ is a solution of problem (1.2) in the case and only in the case when u_3 is a solution of problem (4.5) and (u_1, u_2) is a solution of problem (2.12) for $\dot{z} = u_3$.

If $\mathbf{u} = (u_1, u_2, u_3)^T$, $u_i \in L_2$ is an arbitrary solution of problem (1.1) and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$, $\omega_i \in \mathrm{AC}$ is the corresponding trajectory, then the necessary conditions of Pontryagin's maximum principle must be satisfied for the process (u, ω). Application of the corresponding formulism leads to the relations (for greater detail, see Ref. 1).

$$u_1(t) = \gamma_1(t), \quad u_2(t) = \gamma_2(t), \quad u_3(t) = C\gamma_3(t) \tag{4.6}$$

$$\begin{aligned} &\dot{\omega}_1 = K\omega_2\omega_3 + \gamma_1, \quad \dot{\gamma}_1 = K\gamma_2\omega_3 \\ &\dot{\omega}_2 = -K\omega_1\omega_3 + \gamma_2, \quad \dot{\gamma}_2 = -K\gamma_1\omega_3 \\ &\dot{\omega}_3 = C\gamma_3, \quad \dot{\gamma}_3 = (\gamma_2\omega_1 - \gamma_1\omega_2)K \end{aligned} \tag{4.7}$$

which are satisfied almost everywhere in $[0,T]$.

It follows from Eq. (4.7) that $\gamma_i \in \mathrm{AC}$. Then, by virtue of relations (4.6), the optimal controls u_i can be chosen such that these relations are satisfied for all $t \in [0, T]$. This can be done since, on passing to the equivalent control functions, the magnitude of the criterion does not change. Hence, a solution of initial problem (1.1) also exists in the space AC. Since the right-hand sides of Eq. (4.7) are functions which can be differentiated as many times as desired with respect to each of the arguments ω_i and γ_i, we arrive at a conclusion concerning the existence of derivatives of any order for the chosen smooth solution, that is, $u_i \in C^\infty$.

5. A problem with an unspecified termination time

We will consider a version of initial problem (1.1), assuming that the time T when the process terminates is not specified. In other words, we study the problem of an optimal control of the form

$$\inf_{T>0} \inf_{(u_1, u_2, u_3) \in \mathcal{A}_T} \int_{[0,T]} (u_1^2(t) + u_2^2(t) + C^{-1}u_3^2(t))dt$$

Here the set $\mathcal{A}_T$ has the same meaning as the set $\mathcal{A}$ for a specified termination time T. However, for convenience, the explicit dependence on the parameter T is indicated.

We now construct the minimizing sequence for the chosen problem and, for each natural k, we put

$$T_k = k; \quad u_3^{(k)}(t) = (w_3 - \upsilon_3)/k \text{ when } t \in [0, T_k]$$

We choose the solution of problem (2.2) for $\dot{z} = u_3^{(k)}$ as $u_1^{(k)}, u_2^{(k)} \in L_2([0, T_k])$. Then, from formula (2.12), we have

$$\min_{(u_1, u_2) \in \mathcal{A}_{T_k}(u_3^{(k)})} \int_{[0,T_k]} (u_1^2(t) + u_2^2(t))dt = F(u_3^{(k)})$$

Using representation (2.11), we obtain the inequality

$$F(u_3^{(k)}) = T_k^{-1} \sum_{i=1}^{2} (\tilde{w}_i^{(k)} - \upsilon_i)^2 = k^{-1} \left\| \begin{pmatrix} \tilde{w}_1^{(k)} \\ \tilde{w}_2^{(k)} \end{pmatrix} - \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} \right\|_2^2 \le k^{-1} \left\{ \left\| \begin{pmatrix} \tilde{w}_1^{(k)} \\ \tilde{w}_2^{(k)} \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} \right\|_2 \right\}^2$$

$$\begin{pmatrix} \tilde{w}_1^{(k)} \\ \tilde{w}_2^{(k)} \end{pmatrix} = B^T(T_k) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$B(0) = I, \quad \dot{B} = K\omega_3^{(k)} \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} B, \quad \omega_3^{(k)} = \upsilon_3 + \int_{[0,t]} u_3^{(k)}(\tau)d\tau, \quad t \in [0, T]$$

where $\|\cdot\|_2$ is a Euclidean norm. Since the matrix $B(T_k)$ is orthogonal,

$$\left\|\begin{pmatrix} \tilde{w}_1^{(k)} \\ \tilde{w}_2^{(k)} \end{pmatrix}\right\|_2 = \left\|\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right\|_2$$

and we arrive at the inequality

$$F(u_3^{(k)}) \le k^{-1} d$$

$$d = \left\{\left\|\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}\right\|_2 + \left\|\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix}\right\|_2\right\}^2 \ge 0$$

By construction,

$$(u_1^{(k)}, u_2^{(k)}, u_3^{(k)}) \le \mathcal{A}_{T_k}$$

for all natural k and therefore

$$\inf_{(u_1, u_2, u_3) \in \mathcal{A}_{T_k}} \int\limits_{[0, T_k]} (u_1^2(t) + u_2^2(t) + C^{-1} u_3^2(t))dt \le \int\limits_{[0, T_k]} ((u_1^{(k)}(t))^2 + (u_2^{(k)}(t))^2 + C^{-1}(u_3^{(k)}(t))^2)dt =$$

$$= F(u_3^{(k)})^2 + C^{-1} \int\limits_{[0, T_k]} (u_3^{(k)}(t))^2 dt \le k^{-1}(d + C^{-1}|w_3 - \upsilon_3|^2) \to 0$$

when $k \to \infty$. Consequently,

$$\inf_{T > 0} \inf_{(u_1, u_2, u_3) \in \mathcal{A}_T} \int\limits_{[0, T]} (u_1^2(t) + u_2^2(t) + C^{-1} u_3^2(t))dt = 0$$

The only possible case of free rotation (of uncontrolled angular motion) corresponds to this equality. This manoeuvre is not possible in the case of all boundary conditions. Hence, if the boundary conditions v and w for the angular velocity are such that the angular velocity vector accompanying free rotation cannot pass from position v to position w, then a solution of the optimal control problem of rotation with an unspecified termination time in the class of measurable functions of time does not exist.

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